ON THE DEPRESSION OF AN ALGEBRAIC EQUATION WHEN A PAIR OF ITS ROOTS ARE CONNECTED BY A GIVEN LINEAR RELATION.

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The usual method of depressing $f(x) = 0$, when a given linear relation exists between a and a', two of its roots, is to substitute for x the value of a' (expressed in terms of a); the result, considered as an equation for a, has a common root with $f(a) = 0$, and a can then be found by the greatest common measure method.

Sometimes, however, the following method is to be preferred: Let a and a' be roots of the quadratic $X = x^2 - bx + c = 0$; divide $f(x)$ by X, leaving a remainder of the form $\varphi x + \theta$; to make this vanish identically, equate φ and θ separately to zero, express them as functions of a by the relation $b = a + a'$, $c = aa'$, and the given relation $la + ma' = p$; apply the G. C. M. method to find a the common root of $\varphi = 0$, $\theta = 0$.

In the former method both equations are of the same degree (n suppose), while in the latter φ is of the $(n-1)$st degree in a and θ of the nth.

This will appear from the law of formation of the successive coefficients in the quotient and remainder arising from the division of $f(x)$ by $x^2 - bx + c$; it is evident from the process of division that each coefficient is equal to b times the preceding, minus c times the next preceding, plus the corresponding coefficient in the dividend, until we come to θ, in which case we omit the multiplier b. Now b, when expressed in terms of a, is of the first degree, and c is of the second; hence the second, third, fourth, etc. terms of the quotient will involve a in the first, second, third, etc. degrees; the last, or $(n-1)$st term, will involve a in the $(n-2)$nd degree, φ will be of the $(n-1)$st degree in a, and θ of the nth. The induction in this case is so evident that it need not be formally given.

If the G. C. M. of φ and θ be of the second degree, it will give two values of a, and there are two corresponding values of a', indicating that two pairs of the roots of $f(x) = 0$ are connected by the given relation; similarly if $\varphi = 0$, and $\theta = 0$, have s common roots, there are s pairs of the roots of $f(x) = 0$ connected by the same relation.

Suppose, for example, that we have the equation

$$x^4 - 7x^3 + 8x^2 + 7x + 15 = 0, \tag{1}$$

and that it is known that two of its roots a and a' are connected by the relation $2a' - 3a = 1$.

To solve by the usual method, we substitute $\frac{1}{2}(3a + 1)$ for x in (1); thus

$$27a^4 - 90a^3 - 12a^2 + 82a + 105 = 0; \tag{2}$$

and then find the G. C. M. of (1) and (2).

By the second method

$$\varphi = 65a^3 - 223a^2 + 59a + 75 = 0,$$
$$\theta = 57a^4 - 167a^3 - 5a^2 + 19a - 120 = 0;$$

and then apply the G. C. M. method as usual.

It may be noticed that the second method theoretically lessens the labor of obtaining the G. C. M., since φ is one degree lower than $f(a)$; practically, however, this is not always the case.

There is a case in which the usual method does not assist us in solving a proposed equation, namely, when we have an equation $f(x) = 0$, and it is known that the roots of this equation occur in pairs, and that *each* pair of roots a and a' satisfies the relation $a + a' = r$ (Todhunter, Theory of Equations, Art. 128).

The second method presents no such difficulty.

Suppose, for example, that we have the equation

$$f(x) = x^4 - 12x^3 + 13x^2 + 138x + 112 = 0,$$

where it is known that each pair of roots a and a' satisfies the relation $a + a' = 6$.

By the usual method we should proceed to investigate the common roots of the equations $f(x) = 0$ and $f(6 - x) = 0$. But these equations will be found to coincide completely, and do not afford a solution. The second method furnishes it easily. Write X in the form $x^2 - 6x + c$. $\varphi = 0$ vanishes, and $\theta = 0 = c^2 + 23c + 112$; whence by solving a quadratic equation we obtain c, and the roots are easily found. It may be remarked that when $f(x) = 0$ has equal roots, φ, expressed as a function of a, is the first derivative of $f(a)$.